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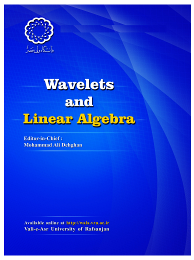

Cyclic wavelet systems in prime dimensional linear vector spaces

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Abstract

Finite affine groups are given by groups of translations and dilations on finite cyclic groups. For cyclic groups of prime order, we develop a time-scale (wavelet) analysis and show that for a large class of non-zero window signals/vectors, the generated full cyclic wavelet system constitutes a frame whose canonical dual is a cyclic wavelet frame.

1. Introduction

Signal processing of discrete, finite length signals is the basis of digital signal processing. Over the last decades, joint time-frequency (time-scale) representations of non-stationary signals have

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achieved significant popularity [6, 10, 14, 15]. Time-frequency (resp. time-scale) representations are obtained by analyzing the signal with respect to an overcomplete function system whose elements are localized in time and in frequency (resp. scale). The obtained data is interpreted using frame theory as introduced by Duffin and Schaeffer in [9]. Among various types of frames, coherent or structured frames such as classic wavelet frames generated by dyadic dilations and integer translations of a window function have been proven to be particularly useful [7]. Similar to wavelet frames, Gabor frames generated by a set of modulations and translations of a given single window function have been studied at length. Coherent frames such as wavelet frames or Gabor frames give us time-frequency (time-scale) representations and redundant time-frequency (time-scale) expansions. Finite frames are smooth and shrewd modeling for any kinds of applications which needs stable redundant time-frequency decomposition such as digital signal processing, image analysis, filter banks, quantization, data analysis and compressed sensing, see [5] and references therein. In compressed sensing, non-localized structured matrices as measurement matrices play significant roles [13, 25, 28].

Commonly used coherent methods are Gabor analysis, wavelet analysis, and wave packet analysis as methods of windowed transforms, see [15, 16, 17, 24, 32] and references therin. The theory of Gabor analysis is based on the modulations and translations of a given window signal (atom). The phase space (time-frequency plane) has a unified group structure, which implies a concrete discretization and quantization. The theory of finite dimensional Gabor analysis from abstract harmonic analysis point of view, is studied in details, see [24] and references therein. From frame theoretical aspects, finite Gabor frames or finite dimensional Gabor systems are precisely quantization of the continuous Gabor transform in the case of the finite Abelian groups [5, 24]. Applications of random finite dimensional Gabor systems (random Gabor matrices) in compressive sensing (CS) problem, have been studied in [24, 25, 26, 27]. Wavelet analysis is a time-scale method in signal processing which is based on the continuous affine group $(0,\infty) \ltimes \mathbb{R}$ as the group of dilations and translation , see [7, 31, 32]. Standard discretization and quantization of the continuous wavelet transform use dyadic dilations and integer translates of the window single (wavelet), see [1, 2, 3] and references therein. Classical methods for wavelet analysis of periodic signals or signals of finite size rely on embedding the vector space of finite signals in the Hilbert space of all complex valued sequences with finite energy ($\|.\|_2$-norm), see [4, 30]. Traditional wavelet analysis of finite dimensional signals or data, are not on finite dimensional analogous to the continuous setting as is the case in Gabor analysis [4, 24, 30]. There is an approach to a wavelet type analysis for signals of finite size as well which has some intersection the approach presented in the following paper. In the series of papers [11, 18, 22], authors presented a finite field approach for analyzing signals in $\mathbb{C}^{\mathbb{F}}$, where $\mathbb{F}$ is a finite field [5].

This article contains 4 sections. Section 2 is devoted to basic notations and a brief summary of Fourier analysis on finite cyclic groups, periodic signal processing, and finite frame theory. In section 3, we present the structure of cyclic dilation groups $\mathbb{U}_p$ and then the finite affine groups $\mathbb{W}_p$. Then we study basic properties of the cyclic wavelet systems. It is shown that for a large class of non-zero window signals (wavelets), the generated cyclic wavelet systems constitute a frame whose canonical dual are cyclic wavelet frames as well.

2. Preliminaries and Notations

Throughout this article we shall use the standard and traditional harmonic analysis modeling for the linear vector space of all periodic signals or finite size data. A given one dimensional (1D) finite discrete data or signal $\mathbf{x}$, i.e. a signal of a given length $N \in \mathbb{N}$ denoted by $\mathbf{x} = [\mathbf{x}(0), ..., \mathbf{x}(N-1)]$, which is a function defined on the set $\{0, ..., N-1\} \subset \mathbb{Z}$. This type of modeling for indexing of finite signals persists us to consider finite signals as functions defined on the group of unit roots of order N, or equivalently as periodic discrete functions (sequences) $\mathbf{x} : \mathbb{Z} \to \mathbb{C}$ with $\mathbf{x}(n + kN) = \mathbf{x}(n)$ for all $0 \leq n \leq N-1$, and $k \in \mathbb{Z}$. With above mathematical modeling of one dimensional (1D) finite discrete signals, the notation $\mathbb{C}^N$ precisely stands for the complex linear vector space of all finite signals of size N. We shall denote the finite additive (Abelian) group of all integers between zero and some non-zero integer number N by $\mathbb{Z}_N$ or $\langle N \rangle$. Roughly speaking, $\mathbb{Z}_N$ contains consists of integers modulo N. The set $\mathbb{Z}_N = \{\overline{0}, \overline{1}, ..., \overline{N-1}\}$ or with a simple notation $\mathbb{Z}_N = \{0, 1, ..., N-1\}$ is a finite cyclic group with respect to the addition module N with the identity element 0 and the additive inverse $N - l$ for the element $l \in \mathbb{Z}_N$. The set of all bijective (i.e. injective and surjective) homomorphisms on $\mathbb{Z}_N$ (i.e. isomorphisms on $\mathbb{Z}_N$) is denoted by $Aut(\mathbb{Z}_N)$ which is a group with respect to composition of automorphisms. For a matrix $\mathbf{X} \in \mathcal{M}_{N\times M}(\mathbb{C})$, the notation : used to denote all elements indexed by a variable. As an example, if $0 \leq n \leq N$ then $\mathbf{X}(n, :)$ is the n-th row of $\mathbf{X}$. This notation coincides with the notations in MATLAB and FORTRAN programming. For a matrix $\mathbf{X} \in \mathcal{M}_{N\times M}(\mathbb{C})$, the Frobenius norm is defined by

$$\|\mathbf{X}\|_{\mathbf{Fr}} = \left(\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} |\mathbf{X}(n,m)|^2\right)^{1/2}. \tag{2.1}$$

The inner product of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ is $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{y}(k)}$, which induces the ℓ^2-norm or the Frobenius norm $\|\mathbf{x}\|_2^2 = \|\mathbf{x}\|_{\mathrm{Fr}}^2 = \langle \mathbf{x}, \mathbf{x}\rangle$ for $\mathbf{x} \in \mathbb{C}^N$. The notation $\|\mathbf{x}\|_0 := |\{k \in \mathbb{Z}_N : \mathbf{x}(k) \neq 0\}|$ counts non-zero entries in $\mathbf{x} \in \mathbb{C}^N$. Let $\ell, l \in \mathbb{Z}_N$. The translation operator $T_l : \mathbb{C}^N \to \mathbb{C}^N$ is $T_l\mathbf{x}(k) = \mathbf{x}(k - l)$ for $\mathbf{x} \in \mathbb{C}^N$ and $l, k \in \mathbb{Z}_N$. The modulation operator $M_\ell : \mathbb{C}^N \to \mathbb{C}^N$ is given by $M_\ell\mathbf{x}(k) = e^{-2\pi i\ell k/N}\mathbf{x}(k)$ for $\mathbf{x} \in \mathbb{C}^N$ and $\ell, k \in \mathbb{Z}_N$. The translation and modulation operators on the Hilbert space $\mathbb{C}^N$ are unitary operators in the Frobenius norm. For $\ell, l \in \mathbb{Z}_N$ we have $(T_l)^* = (T_l)^{-1} = T_{N-l}$ and $(M_\ell)^* = (M_\ell)^{-1} = M_{N-\ell}$. The circular convolution of $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ is defined by

$$\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} \mathbf{x}(l)\mathbf{y}(k - l) \text{ for } k \in \mathbb{Z}_N.$$

In terms of the translation operators we have $\mathbf{x} * \mathbf{y}(k) = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} \mathbf{x}(l)T_l\mathbf{y}(k)$ for $k \in \mathbb{Z}_N$. The circular involution or circular adjoint of $\mathbf{x} \in \mathbb{C}^N$ is given by $\mathbf{x}^*(l) = \overline{\mathbf{x}(-l)} = \overline{\mathbf{x}(N - l)}$. The complex linear space $\mathbb{C}^N$ equipped with the ℓ^1-norm, the circular convolution, and involution is a Banach $*$-algebra. The unitary discrete Fourier Transform (DFT) of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is defined by $\widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{w}_\ell(k)}$, for all $\ell \in \mathbb{Z}_N$ where for all $\ell, k \in \mathbb{Z}_N$ we have $\mathbf{w}_\ell(k) = e^{2\pi i\ell k/N}$. The set $\{\mathbf{w}_\ell : \ell \in \mathbb{Z}_N\}$ is precisely the multiplicative group of all pure frequencies (characters) $\widehat{\mathbb{Z}_N}$ (i.e homomorphisms or characters into the circle group $\mathbb{T}$) on the additive group $\mathbb{Z}_N$. More precisely, the map $\ell \mapsto \mathbf{w}_\ell$ is a group isomorphism between $\mathbb{Z}_N$ and $\widehat{\mathbb{Z}_N}$. Therefore, $\mathbf{w}_{\ell+\ell'} = \mathbf{w}_\ell \mathbf{w}_{\ell'}$ and $\overline{\mathbf{w}_\ell} = \mathbf{w}_{N-\ell}$ for all $\ell, \ell' \in \mathbb{Z}_N$. Thus DFT of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ at the frequency $\ell \in \mathbb{Z}_N$ is

$$\widehat{\mathbf{x}}(\ell) = \mathcal{F}_N(\mathbf{x})(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{w}_\ell(k)} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}(k) e^{-2\pi i \ell k/N}. \tag{2.2}$$

The discrete Fourier transform (DFT) is a unitary transform in the Frobenius norm, i.e. for all $\mathbf{x} \in \mathbb{C}^N$ satisfies the Parseval formula $\|\mathcal{F}_N(\mathbf{x})\|_{\mathbf{Fr}} = \|\mathbf{x}\|_{\mathbf{Fr}}$, which equivalently implies $\sum_{\ell=0}^{N-1} |\widehat{\mathbf{x}}(\ell)|^2 = \sum_{k=0}^{N-1} |\mathbf{x}(k)|^2$. The Polarization identity implies the Plancherel formula $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^N} = \langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle_{\mathbb{C}^N}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$, which equivalently implies $\sum_{\ell=0}^{N-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{\mathbf{y}}(\ell)} = \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{y}(k)}$. The unitary DFT (2.2) satisfies [12, 21]

$$\widehat{T_l\mathbf{x}} = M_l\widehat{\mathbf{x}}, \quad \widehat{M_l\mathbf{x}} = T_{N-l}\widehat{\mathbf{x}}, \quad \widehat{\mathbf{x}^*} = \overline{\widehat{\mathbf{x}}}, \quad \widehat{\mathbf{x} * \mathbf{y}} = \widehat{\mathbf{x}}.\widehat{\mathbf{y}},$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ and $l \in \mathbb{Z}_N$. The inversion discrete Fourier formula (IDFT) for a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is

$$\mathbf{x}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{\mathbf{x}}(k)\mathbf{w}_\ell(k) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{\mathbf{x}}(k) e^{2\pi i \ell k/N}, \quad 0 \le \ell \le N-1. \tag{2.3}$$

A finite sequence $\mathfrak{A} = \{\mathbf{y}_j : 0 \le j \le M-1\} \subset \mathbb{C}^N$ is called a frame (or finite frame) for the finite dimensional complex Hilbert space $\mathbb{C}^N$, if there exist positive constants $0 < A \le B < \infty$ such that

$$A\|\mathbf{x}\|_{\mathrm{Fr}}^2 \le \sum_{j=0}^{M-1} |\langle \mathbf{x}, \mathbf{y}_j \rangle_{\mathbb{C}^N}|^2 \le B\|\mathbf{x}\|_{\mathrm{Fr}}^2, \text{ for all } \mathbf{x} \in \mathbb{C}^N. \tag{2.4}$$

If $\mathfrak{A} = \{\mathbf{y}_j : 0 \le j \le M-1\}$ is a frame for $\mathbb{C}^N$, the synthesis operator $F : \mathbb{C}^M \to \mathbb{C}^N$ is $F\{c_j\}_{j=0}^{M-1} = \sum_{j=0}^{M-1} c_j \mathbf{y}_j$ for all $\{c_j\}_{j=0}^{M-1} \in \mathbb{C}^M$. The adjoint (analysis) operator $F^* : \mathbb{C}^N \to \mathbb{C}^M$ is $F^*\mathbf{x} = \{\langle \mathbf{x}, \mathbf{y}_j \rangle_{\mathbb{C}^N}\}_{j=0}^{M-1}$ for all $\mathbf{x} \in \mathbb{C}^N$. By composing F and F^*, we get the frame operator $S : \mathbb{C}^N \to \mathbb{C}^N$ given by [5]

$$\mathbf{x} \mapsto S\mathbf{x} = FF^*\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle_{\mathbb{C}^N} \mathbf{y}_j \text{ for all } \mathbf{x} \in \mathbb{C}^N, \tag{2.5}$$

which is positive and invertible. In terms of the analysis operator we have $A\|\mathbf{x}\|_{\mathrm{Fr}}^2 \le \|F^*\mathbf{x}\|_{\mathrm{Fr}}^2 \le B\|\mathbf{x}\|_{\mathrm{Fr}}^2$ for $\mathbf{x} \in \mathbb{C}^N$. If $\mathfrak{A}$ is a finite frame for $\mathbb{C}^N$, the set $\mathfrak{A}$ spans the complex Hilbert space $\mathbb{C}^N$ which implies $M \ge N$, where $M = |\mathfrak{A}|$. It should be mentioned that each finite spanning set in $\mathbb{C}^N$ is a finite frame for $\mathbb{C}^N$. The ratio between M and N is called the redundancy of the finite frame $\mathfrak{A}$ (i.e. $\mathrm{red}_{\mathfrak{A}} = M/N$), where $M = |\mathfrak{A}|$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \le j \le M-1\}$ is a finite frame for $\mathbb{C}^N$, each $\mathbf{x} \in \mathbb{C}^N$ satisfies the following reconstruction formulas [5];

$$\mathbf{x} = \sum_{j=0}^{M-1} \langle \mathbf{x}, S^{-1}\mathbf{y}_j \rangle_{\mathbb{C}^N} \mathbf{y}_j = \sum_{j=0}^{M-1} \langle \mathbf{x}, \mathbf{y}_j \rangle_{\mathbb{C}^N} S^{-1}\mathbf{y}_j. \tag{2.6}$$

In this case, the complex numbers $\langle \mathbf{x}, S^{-1}\mathbf{y}_j \rangle_{\mathbb{C}^N}$ are called frame coefficients and the finite sequence $\mathfrak{A}^\bullet := \{S^{-1}\mathbf{y}_j : 0 \le j \le M-1\}$ which is a frame for $\mathbb{C}^N$ as well, is called the canonical dual frame of $\mathfrak{A}$. A finite frame $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ for $\mathbb{C}^N$ is called tight if we have $A = B$. If $\mathfrak{A} = \{\mathbf{y}_j : 0 \leq j \leq M-1\}$ is a tight frame for $\mathbb{C}^N$ with frame bound A, then the canonical dual frame $\mathfrak{A}^\bullet$ is exactly $\{A^{-1}\mathbf{y}_j : 0 \leq j \leq M-1\}$ and for $\mathbf{x} \in \mathbb{C}^M$ we have [5]

$$\mathbf{x} = \frac{1}{A}\sum_{j=0}^{M-1}\langle \mathbf{x}, \mathbf{y}_j\rangle_{\mathbb{C}^N}\mathbf{y}_j. \tag{2.7}$$

3. Cyclic Dilation Operators and Finite Affine Group

In this section we briefly state structure and basic properties of cyclic dilation operators, see [11, 15, 18, 22]. We then present the notion of finite wavelet groups over finite Abelian groups of prime order [23].

Let p be a positive prime integer. The set

$$\mathbb{U}_p := \mathbb{Z}_p - \{0\} = \{1, ..., p-1\}, \tag{3.1}$$

is a finite multiplicative Abelian group of order $p-1$ with respect to the multiplication module p with the identity element 1. The multiplicative inverse for $m \in \mathbb{U}_p$ (i.e. an element $m_p \in \mathbb{U}_p$ with $mm_p \overset{p}{\equiv} m_pm \overset{p}{\equiv} 1$) is m_p which satisfies $m_pm + np = 1$ for some $n \in \mathbb{Z}$, which can be done by Bezout lemma [19, 29].

For $m \in \mathbb{U}_p$, define the **cyclic dilation operator** $D_m : \mathbb{C}^p \to \mathbb{C}^p$ by

$$D_m\mathbf{x}(k) := \mathbf{x}(m_pk)$$

for all $\mathbf{x} \in \mathbb{C}^p$ and $k \in \mathbb{Z}_p$, where m_p is the multiplicative inverse of m in $\mathbb{U}_p$.

In the following propositions we state basic algebraic properties of cyclic dilation operators.

Proposition 3.1. *Let p be a positive prime integer. Then*

1. *For* $(m,k) \in \mathbb{U}_p \times \mathbb{Z}_p$ *we have* $D_mT_k = T_{mk}D_m$.
2. *For* $m, m' \in \mathbb{U}_p$ *we have* $D_{mm'} = D_mD_{m'}$.
3. *For* $(m,k),(m',k') \in \mathbb{U}_p \times \mathbb{Z}_p$ *we have* $T_{k+mk'}D_{mm'} = T_kD_mT_{k'}D_{m'}$.
4. *For* $(m,\ell) \in \mathbb{U}_p \times \mathbb{Z}_p$ *we have* $D_mM_\ell = M_{m_p\ell}D_m$.

Proof. Let $\mathbf{x} \in \mathbb{C}^p$ be given. Then,
(1) For $(m,k) \in \mathbb{U}_p \times \mathbb{Z}_p$ and $s \in \mathbb{Z}_p$ we can write

$$\begin{aligned}
D_mT_k\mathbf{x}(s) &= T_k\mathbf{x}(m_ps) \\
&= \mathbf{x}(m_ps - k) \\
&= \mathbf{x}(m_ps - m_pmk) \\
&= \mathbf{x}(m_p(s - mk)) \\
&= D_m\mathbf{x}(s - mk) = T_{mk}D_m\mathbf{x}(s).
\end{aligned}$$

(2) For $m, m' \in \mathbb{U}_p$ and $s \in \mathbb{Z}_p$ we can write

$$\begin{aligned} D_{mm'}\mathbf{x}(s) &= \mathbf{x}((mm')_p s) \\ &= \mathbf{x}(m'_p m_p s) = D_m D_{m'}\mathbf{x}(s). \end{aligned}$$

(3) It is straightforward from (1) and (2).
(4) For $(m, \ell) \in \mathbb{U}_p \times \mathbb{Z}_p$ and $s \in \mathbb{Z}_p$ we can write

$$\begin{aligned} D_m M_\ell \mathbf{x}(s) &= M_\ell \mathbf{x}(m_p s) \\ &= \overline{\mathbf{w}_\ell(m_p s)}\mathbf{x}(m_p s) \\ &= \overline{\mathbf{w}_{m_p\ell}(s)} D_m\mathbf{x}(s) = M_{m_p\ell} D_m \mathbf{x}(s). \end{aligned}$$

□

The next result states analytic properties of cyclic dilations.

Proposition 3.2. *Let p be a positive prime integer and $m \in \mathbb{U}_p$. Then*

1. *The dilation operator $D_m : \mathbb{C}^p \to \mathbb{C}^p$ is a $*$-homomorphism.*
2. *The dilation operator $D_m : \mathbb{C}^p \to \mathbb{C}^p$ is unitary in $\|.\|_2$-norm and satisfies*

$$(D_m)^* = (D_m)^{-1} = D_{m_p}.$$

3. *For $\mathbf{x} \in \mathbb{C}^p$ we have $\widehat{D_m\mathbf{x}} = D_{m_p}\widehat{\mathbf{x}}$.*

Proof. Let p be a positive prime integer and $m \in \mathbb{U}_p$. Then
(1) If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$, then for all $k \in \mathbb{Z}_p$ we have

$$D_m(\mathbf{x} * \mathbf{y})(k) = \mathbf{x} * \mathbf{y}(m_p k) = \frac{1}{\sqrt{p}} \sum_{s=0}^{p-1} \mathbf{x}(s)\mathbf{y}(m_p k - s).$$

Replacing s with $m_p s$ we get

$$\begin{aligned} \sum_{s=0}^{p-1} \mathbf{x}(s)\mathbf{y}(m_p k - s) &= \sum_{s=0}^{p-1} \mathbf{x}(m_p s)\mathbf{y}(m_p k - m_p s) \\ &= \sum_{s=0}^{p-1} \mathbf{x}(m_p s)\mathbf{y}(m_p(k - s)) \\ &= \sum_{s=0}^{p-1} \mathbf{x}(m_p s) D_m\mathbf{y}(k - s) = \sum_{s=0}^{p-1} D_m\mathbf{x}(s) D_m\mathbf{y}(k - s). \end{aligned}$$

Thus we can write

$$\sum_{s=0}^{p-1} \mathbf{x}(s)\mathbf{y}(m_p k - s) = \sqrt{p}(D_m\mathbf{x}) * (D_m\mathbf{y})(k),$$

which implies $D_m(\mathbf{x} * \mathbf{y}) = (D_m\mathbf{x}) * (D_m\mathbf{y})$. As well as we have

$$(D_m\mathbf{x})^* = D_m\mathbf{x}^*,$$

because we can write

$$\begin{aligned}(D_m\mathbf{x})^*(k) &= \overline{D_m\mathbf{x}(p-k)}\\ &= \overline{\mathbf{x}(m_p(p-k))}\\ &= \overline{\mathbf{x}(m_pp - m_pk)}\\ &= \overline{\mathbf{x}(p - m_pk)}\\ &= \mathbf{x}^*(m_pk) = D_m\mathbf{x}^*(k).\end{aligned}$$

(2) Let $\mathbf{x} \in \mathbb{C}^p$. We will show that $\|D_m\mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$. Thus we can write

$$\begin{aligned}\|D_m\mathbf{x}\|_2^2 &= \sum_{k=0}^{p-1} |D_m\mathbf{x}(k)|^2\\ &= \sum_{k=0}^{p-1} |\mathbf{x}(m_pk)|^2\\ &= \sum_{k=0}^{p-1} |\mathbf{x}(k)|^2 = \|\mathbf{x}\|_2^2.\end{aligned}$$

(3) Let $\mathbf{x} \in \mathbb{C}^p$. For $\ell \in \mathbb{Z}_p$ we have

$$\widehat{D_m\mathbf{x}}(\ell) = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} D_m\mathbf{x}(k)e^{-2\pi i\ell k/p} = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} \mathbf{x}(m_pk)e^{-2\pi i\ell k/p}.$$

Replacing k with mk we achieve

$$\frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} \mathbf{x}(m_pk)e^{-2\pi i\ell k/p} = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} \mathbf{x}(k)e^{-2\pi i\ell mk/p} = D_{m_p}\widehat{\mathbf{x}}(\ell),$$

□

The underlying set

$$\mathbb{U}_p \times \mathbb{Z}_p = \left\{(m,k) : m \in \mathbb{U}_p, k \in \mathbb{Z}_p\right\},$$

equipped with the following group operations

$$(m,k) \ltimes (m',k') := (mm', k + mk'), \quad (m,k)^{-1} := (m_p, m_p.(p-k)), \tag{3.2}$$

is a finite non-Abelian group of order $p(p-1)$ denoted by $\mathbb{W}_p$ and called as **finite affine group** on p integers or the **finite wavelet group** associated to the integer p or simply we call it as the p-**wavelet group**.

The following proposition summarizes basic properties of the finite wavelet group.

Proposition 3.3. *Let $p > 2$ be a positive prime integer. Then*

1. *$\mathbb{W}_p$ is a non-Abelian group of order $p(p-1)$ which contains a copy of $\mathbb{Z}_p$ as a normal Abelian subgroup and a copy of $\mathbb{U}_p$ as a non-normal Abelian subgroup.*
2. *The map $\sigma : \mathbb{W}_p \to \mathcal{U}_{p\times p}(\mathbb{C})$ defined by $(m,k) \mapsto \sigma(m,k) := T_k D_m$ for $(m,k) \in \mathbb{W}_p$, is a unitary representation of the finite affine group $\mathbb{W}_p$ on the finite dimensional Hilbert space $\mathbb{C}^p$.*

Remark 3.4. Cyclic dilations destroy geometric properties of signals and give sculptured rearrangement of signal or data entries. Hence the cyclic dilation matrices are non-localized structured matrices. From compressive sensing view such non-localized structure is beneficial when considered as measurement matrices, while for "dyadic dilations" this makes the traditional discrete wavelet systems [4, 30] practically useless in compressive sensing applications, see [13, 24, 25, 28] and references therein.

4. Cyclic Wavelet Systems in Prime Dimensional Spaces

In this section we study basic properties of cyclic wavelet systems in prime dimensional linear spaces. Throughout this section we assume that p is a positive prime integer.

A **cyclic wavelet system** for the complex Hilbert space $\mathbb{C}^p$ is a family or system of the form

$$\mathcal{W}(\mathbf{y},\Delta) := \left\{\sigma(m,l)\mathbf{y} = T_l D_m \mathbf{y} : (m,l) \in \Delta \subseteq \mathbb{W}_p\right\}, \tag{4.1}$$

for some window signal $\mathbf{y} \in \mathbb{C}^p$ and a subset Δ of $\mathbb{W}_N$. If $\Delta = \mathbb{W}_p$ we put $\mathcal{W}(\mathbf{y}) := \mathcal{W}(\mathbf{y}, \mathbb{W}_N)$, and it is called a **full** cyclic wavelet system. A cyclic wavelet system which spans $\mathbb{C}^p$ is a frame and is referred to as a **cyclic wavelet frame**.

If $\mathbf{y} \in \mathbb{C}^p$ is a window signal then for $\mathbf{x} \in \mathbb{C}^p$ we have

$$\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p} = \langle \mathbf{x}, T_l D_m \mathbf{y}\rangle_{\mathbb{C}^p} = \langle T_{p-l}\mathbf{x}, D_m\mathbf{y}\rangle_{\mathbb{C}^p}, \quad \text{for } (m,l) \in \mathbb{W}_p. \tag{4.2}$$

The matrix of size $(p-1)\times p$ with the entries given by $\{\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p}\}_{(m,l)\in\mathbb{W}_p}$ is called the **cyclic wavelet matrix** and it is denoted by $\mathbf{W}_{\mathbf{y}}^{\mathbf{x}}$, that is for $1 \le m \le p-1$ and $0 \le l \le p-1$ we have

$$(\mathbf{W}_{\mathbf{y}}^{\mathbf{x}})_{ml} = \langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p}$$

The following proposition gives us a Fourier (resp. convolution) representation for the wavelet matrix.

Proposition 4.1. *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ and $(m,l) \in \mathbb{W}_p$. Then,*

1. $\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p} = \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l)$.
2. $\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p} = \mathbf{x} * D_m\mathbf{y}^*(l)$.

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ and $(m, l) \in \mathbb{W}_p$. (1) Using the Plancherel formula we have

$$\begin{aligned}\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p} &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{T_l D_m \mathbf{y}}(\ell)} \\ &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{M_l\widehat{D_m \mathbf{y}}(\ell)} \\ &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)\mathbf{w}_l(\ell)} \\ &= \sum_{\ell=0}^{p-1} \left(\widehat{\mathbf{x}}.\overline{\widehat{D_m \mathbf{y}}}\right)(\ell)\overline{\mathbf{w}_{p-l}(\ell)} = \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l).\end{aligned}$$

(2) Similarly using the Plancherel formula we can write

$$\begin{aligned}\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p} &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)}\mathbf{w}_l(\ell) \\ &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\widehat{(D_m \mathbf{y})^*}(\ell)\mathbf{w}_l(\ell) \\ &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\widehat{(D_m \mathbf{y}^*)}(\ell)\mathbf{w}_l(\ell) \\ &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x} * D_m\mathbf{y}^*}(\ell)\mathbf{w}_l(\ell) = \mathbf{x} * D_m\mathbf{y}^*(l).\end{aligned}$$

□

In the following theorem we present a concrete formulation for the Frobenius norm of the matrix $\mathbf{W}^{\mathbf{x}}_{\mathbf{y}}$.

Theorem 4.2. *Let* $\mathbf{y} \in \mathbb{C}^p$ *be a window signal and* $\mathbf{x} \in \mathbb{C}^p$*. Then,*

$$\|\mathbf{W}^{\mathbf{x}}_{\mathbf{y}}\|^2_{\mathrm{Fr}} = p\left((p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\right). \tag{4.3}$$

Proof. Let $\mathbf{y} \in \mathbb{C}^p$ be a window function, $\mathbf{x} \in \mathbb{C}^p$, and $m \in \mathbb{U}_p$. Using Proposition 4.1 we have

$$\begin{aligned}\sum_{l=0}^{p-1} |\langle \mathbf{x}, \sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p}|^2 &= p\sum_{l=0}^{p-1}\left|\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l)\right|^2 \\ &= p\sum_{l=0}^{p-1}\left|\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(l)\right|^2 \\ &= p\sum_{\ell=0}^{p-1}\left|(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(\ell)\right|^2 = p\sum_{\ell=0}^{p-1}\left|\widehat{\mathbf{x}}(\ell).\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2.\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{m=1}^{p-1}\sum_{l=0}^{p-1}|\langle \mathbf{x},\sigma(m,l)\mathbf{y}\rangle_{\mathbb{C}^p}|^2 &= p\sum_{m=1}^{p-1}\sum_{\ell=0}^{p-1}\left|\widehat{\mathbf{x}}(\ell).\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2 \\
&= p\sum_{m=1}^{p-1}\sum_{\ell=0}^{p-1}\left|\widehat{\mathbf{x}}(\ell)\right|^2\left|\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2 \\
&= p\sum_{\ell=0}^{p-1}\sum_{m=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2|\widehat{D_m\mathbf{y}}(\ell)|^2 \\
&= p\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{D_m\mathbf{y}}(\ell)|^2\right) \\
&= p\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|D_{m_p}\widehat{\mathbf{y}}(\ell)|^2\right) = p\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right).
\end{aligned}$$

Now we can write

$$\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right) = |\widehat{\mathbf{x}}(0)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(0)|^2\right) + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right). \tag{4.4}$$

Replacing m with $m\ell_p$ we have $\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2 = \sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2$, which implies

$$\begin{aligned}
\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right) &= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right) \\
&= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right).
\end{aligned}$$

Hence using (4.4) we get

$$\|\mathbf{W}_{\mathbf{y}}^{\mathbf{x}}\|_{\mathrm{Fr}}^2 = p\left((p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\right).$$

□

As a consequence of (4.3) we present an irreducible decomposition of the unitary representation σ.

Let $\mathcal{B}_p$ be the complex linear subspace in $\mathbb{C}^p$ of dimension $p-1$ which is given by

$$\mathcal{B}_p := \left\{\mathbf{x}\in\mathbb{C}^p : \widehat{\mathbf{x}}(0) = \sum_{k=0}^{p-1}\mathbf{x}(k) = 0\right\}. \tag{4.5}$$

Proposition 4.3. *The linear subspace $\mathcal{B}_p$ is an irreducible subspace of the unitary representation $\sigma : \mathbb{W}_p \to \mathcal{U}_{p\times p}(\mathbb{C})$.*

Proof. It is straightforward to see that $\mathcal{B}_p$ is an invariant subspace of $\mathbb{C}^p$. Let $\mathcal{H}$ be a nontrivial subspace of $\mathcal{B}_p$. It is enough to show that $\mathcal{H}^\perp = \{0\}$. Let $\mathbf{x} \in \mathcal{H}^\perp$ be arbitrary and pick a nonzero vector $\mathbf{y} \in \mathcal{H}$. Using the assumption that $\mathcal{H}$ is an invariant subspace of $\mathcal{B}_p$ we have $\langle \mathbf{x}, T_l D_m \mathbf{y}\rangle = 0$ for all $(m, l) \in \mathbb{U}_p \times \mathbb{Z}_p$. Since $\mathbf{y}$ is a non-zero vector and $\widehat{\mathbf{y}}(0) = 0$ we achieve that $\sum_{\ell=1}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2 \neq 0$. Invoking (4.3) we can write

$$\left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2\right) \|\mathbf{x}\|_{\mathrm{Fr}}^2 = \left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2\right) \|\widehat{\mathbf{x}}\|_{\mathrm{Fr}}^2 = \left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2\right)\left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\right) = \sum_{m=1}^{p-1}\sum_{l=0}^{p-1} |\langle \mathbf{x}, T_l D_m \mathbf{y}\rangle|^2 = 0,$$

which implies that $\mathbf{x} = 0$. □

The following theorem shows that for a large class of non-zero window signals the full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ is a frame for the finite dimensional Hilbert space $\mathbb{C}^p$ with redundancy $p-1$.

Theorem 4.4. *Let $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal. The full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ constitutes a frame for $\mathbb{C}^p$ with the redundancy $p-1$ if and only if $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$.*

Proof. Let $\mathbf{y}$ be a non-zero window signal with $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. Let $0 < A \leq B < \infty$ be given by

$$A := \min\left\{(p-1)\left|\sum_{k=0}^{p-1} \mathbf{y}(k)\right|^2, p\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right\}, \quad B := \max\left\{(p-1)\left|\sum_{k=0}^{p-1} \mathbf{y}(k)\right|^2, p\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right\}. \tag{4.6}$$

If $\mathbf{x} \in \mathbb{C}^p$, then using (4.3) we can write

$$\sum_{m=1}^{p-1}\sum_{l=0}^{p-1} |\langle \mathbf{x}, T_l D_m \mathbf{y}\rangle_{\mathbb{C}^p}|^2 = \|\mathbf{W}_{\mathbf{y}}^{\mathbf{x}}\|_{\mathrm{Fr}}^2 = p\left((p-1)|\widehat{\mathbf{x}}(0)|^2|\widehat{\mathbf{y}}(0)|^2 + \sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right)\right).$$

Thus we achieve

$$\sum_{m=1}^{p-1}\sum_{l=0}^{p-1} |\langle \mathbf{x}, T_l D_m \mathbf{y}\rangle_{\mathbb{C}^p}|^2 = (p-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{k=0}^{p-1} \mathbf{y}(k)\right|^2 + p\left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right). \tag{4.7}$$

Now by (4.6) and equation (4.7) we get

$$\sum_{m=1}^{p-1}\sum_{l=0}^{p-1}|\langle \mathbf{x}, T_lD_m\mathbf{y}\rangle_{\mathbb{C}^p}|^2 = (p-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2 + p\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)$$
$$\leq \max\left\{(p-1)\left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2, p\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right\}\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2 = B\|\mathbf{x}\|_{\mathrm{Fr}}^2.$$

Similarly, by (4.6) and equation (4.7) we achieve

$$\sum_{m=1}^{p-1}\sum_{l=0}^{p-1}|\langle \mathbf{x}, T_lD_m\mathbf{y}\rangle_{\mathbb{C}^p}|^2 = (p-1)|\widehat{\mathbf{x}}(0)|^2\left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2 + p\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)$$
$$\geq \min\left\{(p-1)\left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2, p\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right\}\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2 = A\|\mathbf{x}\|_{\mathrm{Fr}}^2.$$

Conversely, let $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal such that the full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ be a frame for $\mathbb{C}^p$. Then, using (4.7) we get $\widehat{\mathbf{y}}(0) \neq 0$ and $\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2 \neq 0$, which implies that $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. □

Corollary 4.5. *Let $\mathbf{y} \in \mathbb{C}^p$ be a window signal with $\widehat{\mathbf{y}}(0) \neq 0$ and $\|\widehat{\mathbf{y}}\|_0 \geq 2$. Then,*

1. *The full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ is a tight frame for $\mathbb{C}^p$ if and only if $\mathbf{y}$ satisfies $\|\mathbf{y}\|_{\mathbf{Fr}} = \sqrt{p}|\widehat{\mathbf{y}}(0)|$. In this case,*

$$\alpha_{\mathbf{y}} := (p-1)|\sum_{k=0}^{p-1}\mathbf{y}(k)|^2 = p(p-1)|\widehat{\mathbf{y}}(0)|^2 = p\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2, \tag{4.8}$$

   is the frame bound.

2. *The full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ is a tight frame for the Hilbert space $\mathcal{B}_p$ with the frame bound $\beta_{\mathbf{y}}$, where*

$$\beta_{\mathbf{y}} := p\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2 = p\left(\|\mathbf{y}\|_{\mathrm{Fr}}^2 - |\widehat{\mathbf{y}}(0)|^2\right) = p\|\mathbf{y}\|_{\mathrm{Fr}}^2 - \left|\sum_{k=0}^{p-1}\mathbf{y}(k)\right|^2. \tag{4.9}$$

In the theory of frames, with the help of a dual (canonical dual) pair of coherent frames, one can write any function or signal as a superposition of wavelet frame (coherent frame) elements or expansion [5]. The following interesting property of cyclic wavelet frames shows that the canonical dual frame of any full cyclic wavelet frame is again a full cyclic wavelet frame.

Theorem 4.6. *The canonical dual of any full cyclic wavelet frame for $\mathbb{C}^p$ is a full cyclic wavelet frame.*

Proof. Let $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal such that the full cyclic wavelet system $\mathcal{W}(\mathbf{y})$ be a frame for $\mathbb{C}^p$. Let S be the frame operator of $\mathfrak{A} := \mathcal{W}(\mathbf{y})$. We claim that

$$\mathfrak{A}^{\bullet} = \mathcal{W}(\mathbf{y}^{\bullet}) = \{T_l D_m \mathbf{y}^{\bullet} : (m,l) \in \mathbb{W}_p\}, \tag{4.10}$$

where $\mathbf{y}^{\bullet} := S^{-1}\mathbf{y}$. Invoking the group structure of $\mathbb{W}_p$ and since σ is a unitary representation of $\mathbb{W}_p$ we have $T_l D_m S = S T_l D_m$ for all $(m,l) \in \mathbb{W}_p$.

Then we get $S^{-1}T_l D_m = T_l D_m S^{-1}$ for all $(m,l) \in \mathbb{W}_p$ which implies (4.10). □

Corollary 4.7. *The canonical dual of any full cyclic wavelet frame $\mathcal{W}(\mathbf{y})$ with the frame operator S is the full cyclic wavelet frame $\mathcal{W}(S^{-1}\mathbf{y})$ with the frame operator S^{-1}.*

Remark 4.8. The above property of full cyclic wavelet frames (Theorem 4.6) assures that canonical dual of the cyclic wavelet systems is again a cyclic wavelet system. It should be mentioned that a similar property does not hold for traditional wavelet structured frames, for example, canonical dual frames of infinite dimensional wavelet frames are not in general wavelet frame, see [8, 20] and references therein.

Remark 4.9. If p is a prime integer $\mathbb{F} := \mathbb{Z}_p$ is a finite field [23]. It is evident that

$$\mathbb{F}^* = \mathbb{F} - \{0\} = \mathbb{Z}_p - \{0\} = \mathbb{U}_p,$$

and hence the finite affine group $\mathbb{W}_p$ is precisely the group $\mathbb{F}^* \ltimes \mathbb{F}$. It should be also mentioned that Proposition 4.3 and Corollary 4.5 coincides with the direct consequences of results in [11].

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