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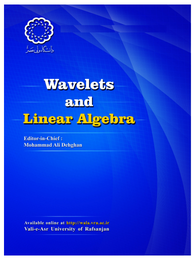

On the characterization of subrepresentations of shearlet group

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Abstract

We regard the shearlet group as a semidirect product group and show that its standard representation is,typically, a quasiregular representation. As a result we can characterize irreducible as well as square-integrable subrepresentations of the shearlet group.

1. Introduction

Among the main tasks of applied harmonic analysis are, firstly, optimally sparse representation of functions belonging to a specific family in terms of some “building blocks” such as frames or

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bases; secondly, identifying discontinuities of functions such as edge detection in image processing. For a long time wavelet theory was the best possible tool for dealing with these problems, but during the last 20 years many alternatives have been suggested. For instance, *steerable pyramid* [18], *two dimensional directional wavelets* [3] and *complex wavelets* [15],[16] were some "directional" versions of wavelets which outperform the wavelets. However, they are not completely satisfying. The first desirable representation was *curvelet* introduced by Candès and Donoho in 2004 [6]. Curvelets have two notable shortcomings: Firstly, this system is not singly generated, to wit, it is not derived from the action of countably many operators applied to a single function; Secondly, there is no algebraic background for curvelets, namely, curvelet transform is not voice transform of some locally compact group. Finally, the best directional representation outperforming curvelets as well as the other representations was *shearlet* that was established in [17]. Kutyniok and et al. usually work with reduced shearlet group, namely, shearlet group with $\mathbb{R}^+$ as the parameter space of dilations and it is known that the standard unitary representation of this group is not square-integrable or even irreducible. In this paper we exhibit two subrepresentations of this representation which are at the same time irreducible and square-integrable.

2. Preliminaries and Notations

Let $A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ be *anisotropic(parabolic) scaling* matrix and $S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ be *shear* matrix acting on the plane. Let $\psi \in L^2(\mathbb{R}^2)$, and for each $a \in \mathbb{R}^+$, $s \in \mathbb{R}$ and $t \in \mathbb{R}^2$ define $\psi_{a,s,t} \in L^2(\mathbb{R}^2)$ by

$$\psi_{a,s,t}(x) = a^{-\frac{3}{4}}\psi(A_a{}^{-1}S_s{}^{-1}(x - t)).$$

Then the *shearlet system* generated by ψ is defined by $\{\psi_{a,s,t} : a \in \mathbb{R}^+, s \in \mathbb{R}, t \in \mathbb{R}^2\}$. The associated *continuous shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is given by

$$\mathcal{SH}_\psi f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{C}$$

$$\mathcal{SH}_\psi f(a, s, t) = \langle f, \psi_{a,s,t}\rangle.$$

Consider the *shearlet group* $\mathbb{S}$, defined as the set $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ along with the multiplication law given by

$$(a, s, t)(a', s', t') = (aa', s + \sqrt{a}s', t + S_s A_a t').$$

The shearlet group is a locally compact group and as mentioned in [8], the left and righ Haar measures of this group, respectively, are

$$d\mu_l(a, s, t) = \frac{da}{a^3}dsdt, \quad d\mu_r(a, s, t) = \frac{da}{a}dsdt.$$

Moreover, $\sigma : \mathbb{S} \longrightarrow \mathcal{U}(L^2(\mathbb{R}^2))$ defined by

$$\sigma(a, s, t)\psi = \psi_{a,s,t}$$

is a unitary representation of $\mathbb{S}$ on $L^2(\mathbb{R}^2)$.
A function $\psi \in L^2(\mathbb{R}^2)$ is called a *continuous shearlet*, if it satisfies the admissibility condition. That is, $\|\mathcal{SH}_\psi f\|^2_{L^2(\mathbb{S})} < \infty, \quad$ for any $f \in L^2(\mathbb{R}^2)$.
As it is shown in [8], the shearlet transform tends to be an isometry. Indeed, for $\psi \in L^2(\mathbb{R}^2)$ consider the following two quantities

$$C^+_\psi = \int_0^\infty \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{{\xi_1}^2} d\xi_2 d\xi_1, \qquad C^-_\psi = \int_{-\infty}^0 \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{{\xi_1}^2} d\xi_2 d\xi_1.$$

For any $f \in L^2(\mathbb{R}^2)$, we have

$$\begin{aligned} \|\mathcal{SH}_\psi f\|^2_{L^2(\mathbb{S})} = \int_{\mathbb{S}} |\langle f, \psi_{a,s,t}\rangle|^2 \frac{da}{a^3} ds dt &= C^+_\psi \int_{\mathbb{R}} \int_0^\infty |\widehat{f}(\omega)|^2 d\omega_1 d\omega_2 \\ &+ C^-_\psi \int_{\mathbb{R}} \int_{-\infty}^0 |\widehat{f}(\omega)|^2 d\omega_1 d\omega_2. \end{aligned} \tag{*}$$

So if $C^+_\psi = C^-_\psi = 1$, then the shearlet transform is an isometry.
According to [8], for any *classical shearlet* we have $C^+_\psi = C^-_\psi = 1$, where by classical shearlet we mean any $\psi \in L^2(\mathbb{R}^2)$ which is wavelet-like along one axis and bump-like along another one. As an example consider ψ satisfying the following condition:

$$\widehat{\psi}(\xi) = \widehat{\psi}(\xi_1, \xi_2) = \widehat{\psi_1}(\xi_1)\widehat{\psi_2}(\frac{\xi_2}{\xi_1}),$$

where $\psi_1 \in L^2(\mathbb{R})$ is a discrete wavelet in the sense that it satisfies the discrete Calderón condition, given by

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi_1}(2^{-j}\eta)|^2 = 1, \text{for a.e.} \quad \eta \in \mathbb{R}$$

with $\widehat{\psi_1} \in C^\infty(\mathbb{R})$ and $supp(\widehat{\psi_1}) \subseteq [-\frac{1}{2}, -\frac{1}{16}] \bigcup [\frac{1}{16}, \frac{1}{2}]$, and $\psi_2 \in L^2(\mathbb{R})$ is a bump function in the sense that

$$\sum_{k=-1}^{1} |\widehat{\psi_2}(\gamma + k)|^2 = 1, \text{for a.e.} \quad \gamma \in [-1, 1]$$

where $\widehat{\psi_2} \in C^\infty(\mathbb{R})$ and $supp(\widehat{\psi_2}) \subseteq [-1, 1]$.

Now (*) shows that ψ is admissible if

$$\int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\xi)|^2}{{\xi_1}^2} d\xi < \infty.$$

3. Main results

Consider the semidirect product $H \times_\tau K$ of locally compact groups H and K, with the operations:

$$(h, k)(h', k') = (hh', k\tau_h(k'))$$

$$(h,k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})),$$

in which, τ is supposed to be a homomorphism from H to the set of automorphisms of K. Following [4], the quasiregular representation$(U, L^2(K))$ of $G = H \times_\tau K$, which in general is not irreducible, is defined by

$$U(h,k)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yk^{-1})), \qquad (h,k) \in G$$

in which, $f \in L^2(K)$ and $y \in K$.
From now on assume that K is also Abelian and consider $\widehat{K}$ as the dual group of K and denote its left Haar measure by $d\omega$. Then one can define a continuous action from H on $\widehat{K}$ by $(h,\omega) \mapsto \omega o \tau_{h^{-1}}$. This action will play an important role in our discussion. Now for a fix $\omega \in \widehat{K}$ the stabilizer and the orbit of ω, that play a key role in our discussion are defined respectively by

$$H^\omega := \{h \in H; \omega o \tau_{h^{-1}} = \omega\}, \quad O_\omega := \{\omega o \tau_{h^{-1}}; h \in H\}.$$

H^ω is a closed subgroup of H and O_ω is an $H-$invariant subset in $\widehat{K}$.
For any measurable subset A of $\widehat{K}$ of positive measure, put

$$L^2_A(K) := \{\psi \in L^2(K); supp(\widehat{\psi}) \subseteq A\}.$$

It is easy to show that $L^2_A(K)$ is a translation invariant closed subspace of $L^2(K)$.
Here we quote some essential theorems [4, Theorems 2.6 and 2.9] by which we deduce our main results:

Proposition 3.1. *With the notations as above, and for a measurable subset A in $\widehat{K}$ we have:*
i) $L^2_A(K)$ are the only translation invariant closed subspaces of $L^2(K)$;
ii) The closed subspaces $L^2_A(K)$ are $U-$invariant if A is an invariant subset in $\widehat{K}$ with respect to the action of H on $\widehat{K}$.
(So in this case the restriction of U on $L^2_A(K)$ denoted by U_A is a subrepresentation of U).

An H-invariant measurable subset A of $\widehat{K}$ is called *ergodic* if every invariant subset of A is null or conull(complement of a null set). For instance, orbits are ergodic subsets.

Corollary 3.2. *A nonzero closed subspace M of $L^2(K)$ is invariant under the representation U if and only if $M = L^2_A(K)$ for some measurable H-invariant subset A of $\widehat{K}$ of positive measure. Moreover the subrepresentation U_A is irreducible if and only if A is ergodic.*

Proposition 3.3. *Let $G = H \times_\tau K$ be the semidirect product of H and K. If $A \subseteq \widehat{K}$ is an ergodic set of positive measure such that $A = O_\omega$ a.e. for some ω, then:*
i) The representation U_A is square-integrable if and only if H^ω is compact;
ii) $\psi \in L^2_A(K)$ is admissible if and only if $h \mapsto \widehat{\psi}(\gamma o \tau_h)$ is in $L^2(H)$ for almost all $\gamma \in A$.

The special case $K = R^n$ has already been investigated in [2, chapter 9] , and [12, corollary 5.24] .

Now we are able to apply the theory to our favorite group, the shearlet group. So we regard the shearlet group as the semidirect product group

$$\mathbb{S} = (\mathbb{R}^+ \times_\tau \mathbb{R}) \times_\lambda \mathbb{R}^2$$

in which, τ and λ are given by $\tau_a(s) = \sqrt{a}s$, and $\lambda_{(a,s)}(t) = S_s A_a t$.
We find out that the standard representation of the shearlet group, namely, $\sigma(a, s, t)\psi = \psi_{a,s,t}$ is the quasiregular representation of this semidirect product group. Indeed

$$\begin{aligned}\sigma(a, s, t)\psi(x) &= a^{-\frac{3}{4}}\psi(A_a^{-1}S_s^{-1}(x - t))\\ &= a^{-\frac{3}{4}}\psi(S_{\tau_{\frac{1}{a}}(-s)}A_{\frac{1}{a}}(x - t))\\ &= a^{-\frac{3}{4}}\psi(\lambda_{(\frac{1}{a},\tau_{\frac{1}{a}}(-s))}(x - t))\\ &= \delta(a, s)^{\frac{1}{2}}\psi(\lambda_{(a,s)^{-1}}(x - t)),\end{aligned}$$

where $\delta(a, s)$ is given by:

$$\begin{aligned}d\mu_{\mathbb{R}^2}(t) &= \delta(a, s)d\mu_{\mathbb{R}^2}(\lambda_{(a,s)}(t))\\ &= \delta(a, s)d\mu_{\mathbb{R}^2}(S_s A_a t)\\ &= \delta(a, s)|\det S_s A_a|d\mu_{\mathbb{R}^2}(t) && \text{(by [11, Theorem 2.44])}\\ &= \delta(a, s)a^{\frac{3}{2}}d\mu_{\mathbb{R}^2}(t).\end{aligned}$$

Here it is natural to ask that if Kutyniok and et al. did not regard the family $\psi_{a,s,t}$ as quasiregular representation, then what was their approach? The path from wavelet to shearlet was "*wavelets with composite dilations*". Indeed seeking for more flexibility, authors in [13], inserted one additional dilation operator in the affine system to produce systems like:

$$\psi_{A,B,k} = \{D_A D_B T_k\psi; A, B \in GL_2(\mathbb{R}), k \in \mathbb{R}^2\},$$

in which, $T_k\psi(x) = \psi(x - k)$ and $D_A\psi(x) = |det(A)|^{-\frac{1}{2}}\psi(A^{-1}x)$.
From this point of view, shearlet systems are special cases of composite dilation wavelets in which, the matrices A and B are taken as:

$$A := A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}, a \in \mathbb{R}^+, \quad B := S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, s \in \mathbb{R}.$$

Now we are ready to exhibit our main theorem. But first note that for a unitary representation π of the locally compact group G on the Hilbert space $\mathcal{H}$ and an invariant closed subspace $\mathcal{M}$ of $\mathcal{H}$, it is well known that $\mathcal{M}^\perp$ is also an invariant closed subspace of $\mathcal{H}$ and hence π is a direct sum of two subrepresentations resulting from $\mathcal{M}$ and $\mathcal{M}^\perp$. The following lemma is needed in our main theorem (3.5)

Lemma 3.4. *The action of* $(\mathbb{R}^+ \times_\tau \mathbb{R})$ *on* $\widehat{\mathbb{R}^2} = \mathbb{R}^2$ *given by*

$$(a, s).\gamma := \gamma o \lambda_{(a,s)^{-1}}$$

is free. That is, for any $\gamma \in \widehat{\mathbb{R}^2} = \mathbb{R}^2$ *we have* $H^\gamma = \{(1, 0)\}$.

Proof. Since

$$\gamma o \lambda_{(a,s)^{-1}}(t) = \gamma(S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t) = e^{2\pi i \gamma . S_{\frac{-s}{\sqrt{a}}} A_{\frac{1}{a}} t} = e^{2\pi i A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt{a}}} \gamma . t} = A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt{a}}} \gamma(t)$$

we have

$$\begin{aligned}(a, s).\gamma := \gamma o \lambda_{(a,s)^{-1}} &= A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt{a}}} \gamma \\ &= \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-s}{\sqrt{a}} & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a} & 0 \\ \frac{-s}{a} & \frac{1}{\sqrt{a}} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{a}\gamma_1 \\ \frac{-s}{a}\gamma_1 + \frac{1}{\sqrt{a}}\gamma_2 \end{pmatrix}.\end{aligned}$$

Now $\begin{pmatrix} \frac{1}{a}\gamma_1 \\ \frac{-s}{a}\gamma_1 + \frac{1}{\sqrt{a}}\gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$ yields $(a, s) = (1, 0)$. □

In particular, since $(a, s).\gamma = \begin{pmatrix} \frac{1}{a}\gamma_1 \\ \frac{-s}{a}\gamma_1 + \frac{1}{\sqrt{a}}\gamma_2 \end{pmatrix}$, we deduce this action has five orbits as follows:

$$O_{(0,0)} = \{(0, 0)\}$$

$$O_{(0,\gamma_2)} = \{(0, y) \in \mathbb{R}^2 : y > 0\}, \gamma_2 > 0$$

$$O_{(0,\gamma_2)} = \{(0, y) \in \mathbb{R}^2 : y < 0\}, \gamma_2 < 0$$

$$O_{(\gamma_1,\gamma_2)} = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \gamma_1 > 0$$

$$O_{(\gamma_1,\gamma_2)} = \{(x, y) \in \mathbb{R}^2 : x < 0\}, \gamma_1 < 0.$$

Computations are straightforward, for example, for $\gamma_2 > 0$ we have:

$$O_{(0,\gamma_2)} = \{(0, \frac{1}{\sqrt{a}}\gamma_2) \in \mathbb{R}^2 : a > 0\} = \{(0, y) \in \mathbb{R}^2 : y > 0\}.$$

Defining $A_+ := O_{(1,0)}$ and $A_- := O_{(-1,0)}$, we have:

Theorem 3.5. *Irreducible as well as square-integrable subrepresentations of the shearlet group are precisely the following two:*

$$\sigma_+ : \mathbb{S} \longrightarrow \mathcal{U}(L^2_{A_+}(\mathbb{R}^2)), \qquad \sigma(a,s,t)\psi = \psi_{a,s,t}$$

$$\sigma_- : \mathbb{S} \longrightarrow \mathcal{U}(L^2_{A_-}(\mathbb{R}^2)), \qquad \sigma(a,s,t)\phi = \phi_{a,s,t}$$

Proof. Since the action defined on $\widehat{\mathbb{R}^2}$ is free so for any $\omega \in \widehat{\mathbb{R}^2}$, H^ω is compact. Moreover A_+ and A_- are orbits of positive measure, thus proposition 3.3 (i) shows the above representations are square-integrable. On the other hand, A_+ and A_- are the only invariant subsets of positive measure in $\mathbb{R}^2$, hence corollary 3.2 concludes the proof. □

Corollary 3.6. *With notations as in theorem (3.5), the standard representation of the shearlet group is direct sum of two irreducible representations, in fact, we have*

$$\sigma = \sigma_+ \oplus \sigma_-$$

Proof. It is sufficient to show that $[L^2_{A_+}(\mathbb{R}^2)]^\perp = L^2_{A_-}(\mathbb{R}^2)$. Indeed for $f \in L^2_{A_-}(\mathbb{R}^2), g \in L^2_{A_+}(\mathbb{R}^2)$, $\widehat{f}$ and $\widehat{g}$ vanish on A_+ and A_- , respectively. So we have $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle = \int_{\widehat{\mathbb{R}^2}} \widehat{f}\overline{\widehat{g}} = \int_{A_-\cup A_+} \widehat{f}\overline{\widehat{g}} = 0$ which implies $f \in [L^2_{A_+}(\mathbb{R}^2)]^\perp$. On the other hand, $f \in [L^2_{A_+}(\mathbb{R}^2)]^\perp$ implies that , $\int_{A_+} \widehat{f}\overline{\widehat{g}} = \langle f, g\rangle = 0$ for any $g \in L^2_{A_+}(\mathbb{R}^2)$. Now since $L^2_{A_+}(\mathbb{R}^2)$ is a selfdual Banach space we may regard $\widehat{f}$ as a linear functional on $L^2_{A_+}(\mathbb{R}^2)$ and deduce $\widehat{f}$ vanishes on A_+. So $f \in L^2_{A_-}(\mathbb{R}^2)$. □

At this point, we are able to show the irreducibility of the standard representation of *full shearlet group* which is studied for example in [9]. Full shearlet group is the set $\mathbb{R}^* \times \mathbb{R} \times \mathbb{R}^2$ along with the multiplication law given by

$$(a, s, t)(a', s', t') = (aa', s + \sqrt{|a|}s', t + S_s A_a t'),$$

in which dilation and shear matrices are given by

$$A_a = \begin{pmatrix} a & 0 \\ 0 & sgn(a)\sqrt{|a|} \end{pmatrix}, \qquad S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Moreover, we have $\tau_a(s) = \sqrt{|a|}s$, and $\lambda_{(a,s)}(t) = S_s A_a(t)$. So the action reads

$$(a, s).\gamma := \gamma o \lambda_{(a,s)^{-1}} = A_{\frac{1}{a}} S^T_{\frac{-s}{\sqrt{|a|}}}\gamma = \begin{pmatrix} \frac{1}{a}\gamma_1 \\ \frac{-sgn(a)s}{|a|}\gamma_1 + \frac{sgn(a)}{\sqrt{|a|}}\gamma_2 \end{pmatrix}.$$

Now it is easy to see that this action has three orbits:

$$O_{(0,0)} = \{(0, 0)\}$$

$$O_{(0,\gamma_2)} = \{(0, y) \in \mathbb{R}^2 : y \neq 0\}, \gamma_2 \neq 0$$

$$O_{(\gamma_1,\gamma_2)} = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}, \gamma_1 \neq 0.$$

Since there is only one positive measure orbit (almost equal to the plane), so as it is shown in [9], the representation of full shearlet group is irreducible.

As a result of our approach we offer an alternative proof for the well known admissibility condition:

Proposition 3.7. *A function ψ belonging to $L^2_{A_+}(\mathbb{R}^2)$ or $L^2_{A_-}(\mathbb{R}^2)$ is admissible if and only if*

$$\int_{\mathbb{R}^2} \frac{|\widehat{\psi}(\lambda,\eta)|^2}{\lambda^2} d\lambda d\eta < \infty.$$

Proof. For any $\gamma \in \widehat{\mathbb{R}^2} = \mathbb{R}^2$ we have

$$\begin{aligned}
\int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(\gamma o \lambda_{(a,s)})|^2 ds \frac{da}{a^{\frac{3}{2}}} &= \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(A_a S_s^T \gamma)|^2 ds \frac{da}{a^{\frac{3}{2}}} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(A_a(\gamma_1, s\gamma_1+\gamma_2))|^2 ds \frac{da}{a^{\frac{3}{2}}} \\
(s\gamma_1+\gamma_2 =: t \Rightarrow \gamma_1 ds = dt) & \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(A_a(\gamma_1, t))|^2 \frac{dt}{\gamma_1}\frac{da}{a^{\frac{3}{2}}} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(a\gamma_1,\ \sqrt{a}t)|^2 \frac{dt}{\gamma_1}\frac{da}{a^{\frac{3}{2}}} \\
&= \int_{\mathbb{R}^+}\int_{\mathbb{R}} |\widehat{\psi}(a\gamma_1,\ \sqrt{a}t)|^2 \frac{da}{a^{\frac{3}{2}}}\frac{dt}{\gamma_1} \\
(a\gamma_1 =: \lambda \Rightarrow da = \frac{d\lambda}{\gamma_1},\ \sqrt{a} = \frac{\sqrt{\lambda}}{\sqrt{\gamma_1}}, \frac{1}{a^{\frac{3}{2}}} = \frac{\gamma_1^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}}) & \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\lambda, \frac{\sqrt{\lambda}}{\sqrt{\gamma_1}}t)|^2 \frac{d\lambda}{\lambda^{\frac{3}{2}}}\frac{dt}{\sqrt{\gamma_1}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\lambda, \frac{\sqrt{\lambda}}{\sqrt{\gamma_1}}t)|^2 \frac{dt}{\sqrt{\gamma_1}}\frac{d\lambda}{\lambda^{\frac{3}{2}}} \\
(\frac{\sqrt{\lambda}}{\sqrt{\gamma_1}}t =: \eta \Rightarrow \frac{\sqrt{\lambda}}{\sqrt{\gamma_1}}dt =: d\eta) & \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\lambda,\eta)|^2 \frac{d\eta}{\sqrt{\lambda}}\frac{d\lambda}{\lambda^{\frac{3}{2}}} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\widehat{\psi}(\lambda,\eta)|^2 d\eta \frac{d\lambda}{\lambda^2}.
\end{aligned}$$

Now proposition 3.3 (ii) concludes the proof. □

4. conclusion

After the introduction of shearlets relative to bivariate signals, the theory was extended by the authors of [10], to multivariate signals. Besides, due to rich group-theoretical background, shearlets got immediately popular so that many mathematicians approached the subject from various aspects. For example, in [7] the shearlet group was considered as an extension of the Heisenberg group and recently authors in [1] studied the shearlet group as a reproducing subgroup of the symplectic group $Sp(2,\mathbb{R})$. But similar to any theory in the scope of harmonic analysis, shearlet theory is supposed to be extended to the setting of locally compact groups. Among the advantages of our approach, is the possibility of extending shearlet theory from Euclidean space to locally compact groups. In a forthcoming paper [14], we will investigate this issue.

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